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ON TWO KINDS OF ELLIPSOIDAL INHOMOGENEITIES IN AN INFINITE ELASTIC BODY: AN APPLICATION TO A HYBRID COMPOSITE†

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Abstract—The problem of two kinds of ellipsoidal inhomogeneities embedded in an elastic body is formulated with an application to a hybrid (three-phase) composite. The analytical tool used in this study is a combination of Eshelby's equivalent inclusion method [1] and Mori–Tanaka's back stress analysis [2], and therefore the results are valid for large volume fraction of inhomogeneities. As a demonstration, two types of hybrid composites are examined: (i) fiber–fiber; and (ii) fiber–particulate systems.

1. INTRODUCTION

When an ellipsoidal inhomogeneity is embedded in an infinite elastic body, several related problems can be solved rather simply by Eshelby's equivalent inclusion method [1]. For example, the overall stiffness of a two-phase composite material can be easily computed once the corresponding eigenstrain is solved by this method.

However, when the volume fraction of inhomogeneity (filler) becomes large, Eshelby's equivalent inclusion method must be modified so as to take into account the interaction among inhomogeneities as well as that between an inhomogeneity and the outer boundary of the composite. The effect of the above interactions have been known as "a back stress" to material scientists. Mori and Tanaka [2] discussed such a problem within the framework of Eshelby's equivalent inclusion method. In Mori–Tanaka's paper only one kind of inclusion was treated.

Recently Taya and Mura [3] have applied Mori–Tanaka's method to a penny-shaped crack at a fiber end in a short-fiber reinforced composite to compute the energy release rate of the fiber end crack and the weakened Young's modulus of the composite. In their model one kind of inhomogeneity (fiber) was treated by Mori–Tanaka's back stress analysis.

Here we extend Mori–Tanaka's back stress analysis to hybrid (three-phase) composites where two kinds of inhomogeneities are embedded in an infinite elastic body in order to obtain the overall stiffness of the composite. To compute the overall stiffness of a hybrid composite, one can also use a "self-consistent method" [4, 5]. However, it requires a numerical computation and also gives rise to inaccurate results when the stiffness of the constituent phases differ from one another to a great extent, e.g. the case of soft matrix-rigid fiber, or fiber-crack system. On the other hand, the present formulation gives us closed form results for the overall stiffness. Hence the computation is simply a parametric one.

We first describe a theoretical formulation and then apply it to two types of hybrid composites: (i) fiber–fiber; and (ii) fiber-particulate systems.

2. FORMULATION

Consider an infinite elastic body which contains infinite number of two kinds of inhomogeneities and is subjected to the applied stress σ_{ij}^0 as shown in Fig. 1. For later conveniences, the domains of two kinds of ellipsoidal inhomogeneties are denoted by Ω_1 and Ω_2 and that of the infinite body is denoted by D. Hence the domain of the matrix is D-Ω_1-Ω_2. Note that Ω_1 can represent a particular inhomogeneity of type 1 or all inhomogeneities of type 1. This is also the case with Ω_2. Let the elastic constants of the matrix, and inhomogeneities Ω_1 and Ω_2 be C_{ijkl}^0, C_{ijkl}^1 and C_{ijkl}^2, respectively. The volume fractions of Ω_1 and Ω_2 are denoted by f_1 and f_2, respectively.

We assume in this paper that all inhomogeneities are aligned in the uniaxial loading direction

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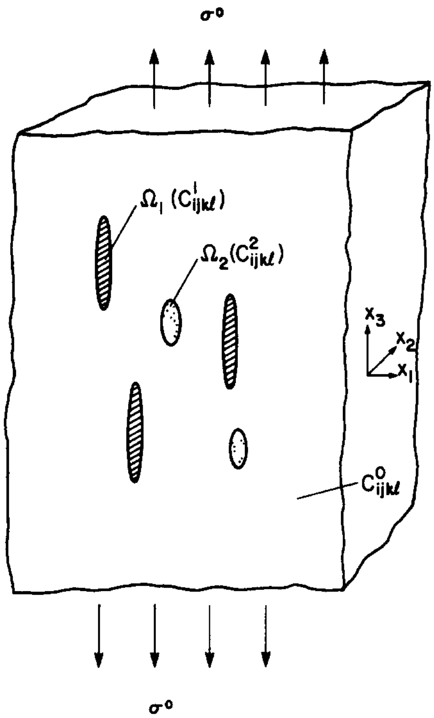

Fig. 1. A calculation model

(say x_3-axis). It should be noted that our formulation below is applicable to more general cases of geometry. Further we assume that the three-phases (matrix, Ω_1 and Ω_2) are linearly elastic and isotropic.

Under the applied stress σ^0_{ij} the average of the total stress in the matrix is given by $\sigma^0_{ij} + \langle\sigma_{ij}\rangle_M$ [2] and

$$\langle\sigma_{ij}\rangle_M = C^0_{ijkl}\tilde{\epsilon}_{kl} \tag{1}$$

where $\tilde{\epsilon}_{kl}$ is the average strain disturbance due to all Ω_1 and Ω_2. Now introduce a single inhomogeneity of type 1 (Ω_1) into the composite, then the equivalent inclusion method yields in D

$$\begin{aligned}\sigma^0_{ij} + \sigma^1_{ij} &= C^0_{ijkl}(\epsilon^0_{kl} + \tilde{\epsilon}_{kl} + \epsilon^1_{kl} - \epsilon^*_{kl})\\ &= C^1_{ijkl}(\epsilon^0_{kl} + \tilde{\epsilon}_{kl} + \epsilon^1_{kl})\end{aligned} \tag{2}$$

where σ^1_{ij} and ϵ^1_{ij} are the disturbance of the stress and strain due to this single Ω_1, respectively. ϵ^*_{ij} is the corresponding eigenstrain which has non-vanishing components in the domain of this single Ω_1 and becomes zero outside this single Ω_1. For the entire domain D the following relation always holds;

$$\sigma^0_{ij} = C^0_{ijkl}\epsilon^0_{kl}. \tag{3}$$

With eqn (3), eqn (2) yields

$$\sigma^1_{ij} = C^0_{ijkl}(\tilde{\epsilon}_{kl} + \epsilon^1_{kl} - \epsilon^*_{kl}). \tag{4}$$

Following Eshelby[1], we have

$$\epsilon^1_{kl} = S^1_{klmn}\epsilon^*_{mn} \quad \text{in } \Omega_1 \tag{5}$$

where S^1_{klmn} is the Eshelby's tensor which depends on C^0_{ijkl} and the geometry of Ω_1. Since the added single inhomogeneity Ω_1 can represent any single Ω_1, the domain Ω_1 in the preceding equations is meant for any inhomogeneity of type 1.

Next we add another single inhomogeneity of type 2 (Ω_2) to this composite system D. Then we have in D

$$\sigma^0_{ij} + \sigma^2_{ij} = C^0_{ijkl}(\epsilon^0_{kl} + \tilde{\epsilon}_{kl} + \epsilon^2_{kl} - \epsilon^{**}_{kl})$$
$$= C^2_{ijkl}(\epsilon^0_{kl} + \tilde{\epsilon}_{kl} + \epsilon^2_{kl}). \tag{6}$$

With eqn (2), eqn (6) provides

$$\sigma^2_{ij} = C^0_{ijkl}(\tilde{\epsilon}_{kl} + \epsilon^2_{kl} - \epsilon^{**}_{kl}) \tag{7}$$

ϵ^2_{kl} is again related to ϵ^{**}_{mn} as

$$\epsilon^2_{kl} = S^2_{klmn}\epsilon^{**}_{mn} \quad \text{in } \Omega_2 \tag{8}$$

where ϵ^{**}_{ij} is the eigenstrain defined in Ω_2, and S^2_{klmn} depends on C^0_{ijkl} and the geometry of inhomogeneity of type 2. Since the disturbed stress σ_{ij} must satisfy $\int_D \sigma_{ij}\,\mathrm{d}V = 0$, we obtain

$$(1 - f_1 - f_2)\langle\sigma_{ij}\rangle_M + f_1\langle\sigma^1_{ij}\rangle + f_2\langle\sigma^2_{ij}\rangle = 0 \tag{9}$$

where $\langle\ \rangle$ denotes the volume averaged quantity.

Eliminating ϵ^1_{ij} and ϵ^2_{ij} through eqns (5) and (8), we have three unknowns, i.e. $\tilde{\epsilon}_{ij}$, ϵ^*_{ij} and ϵ^{**}_{ij}, which will be solved by the three equations (eqns (2), (6) and (9)). Once ϵ^*_{ij} and ϵ^{**}_{ij} are solved, we can compute the overall stiffness of the composite by using the equivalence of the strain energies:

$$\frac{1}{2}C^{c\,-1}_{ijkl}\sigma^0_{ij}\sigma^0_{kl} = \frac{1}{2}C^{0\,-1}_{ijkl}\sigma^0_{ij}\sigma^0_{kl} + \frac{1}{2}f_1\sigma^0_{ij}\epsilon^*_{ij}$$
$$+ \frac{1}{2}f_2\sigma^0_{ij}\epsilon^{**}_{ij} \tag{10}$$

where $C^{0\,-1}_{ijkl}$ and $C^{c\,-1}_{ijkl}$ are the compliances of the matrix and the composite, respectively. The details of the derivation of eqn (10) are given in Appendix A.

The computation of the eigenstrains and the overall stiffness of the composite are described below for two types of hybrid composites: (i) fiber–fiber; and (ii) fiber-particulate systems. Let the inhomogeneity of type 1 be fiber-1, and the inhomogeneity of type 2 be fiber-2 (fiber–fiber system) or particulate (fiber-particulate system).

2.1 *Computation of* ϵ^*_{ij} *and* ϵ^{**}_{ij}

Referring to Fig. 1, the non-vanishing applied stress is σ^0_{33} (denoted by σ^0) and all fibers are aligned along the loading axis (x_3-axis). Hence the non-vanishing components of ϵ^*_{ij}, ϵ^{**}_{ij} and $\tilde{\epsilon}_{ij}$ are $ij = 11$, 22 and 33. It is also noted that the system of Fig. 1 gives rise to a transverse insotropy, i.e. $\epsilon^*_{11} = \epsilon^*_{22}$, $\epsilon^{**}_{11} = \epsilon^{**}_{22}$ and $\tilde{\epsilon}_{11} = \tilde{\epsilon}_{22}$.

In setting $ij = 11$ and 33 in eqn (2), we obtain

$$C^*_{11}\epsilon^*_{11} + C^*_{12}\epsilon^*_{33} = -2D_1^*(\epsilon^0_{11} + \tilde{\epsilon}_{11}) - (\epsilon^0_{33} + \tilde{\epsilon}_{33}) \tag{11}$$

$$C^*_{21}\epsilon^*_{11} + C^*_{22}\epsilon^*_{33} = -2(\epsilon^0_{11} + \tilde{\epsilon}_{11}) - D_2^*(\epsilon^0_{33} + \tilde{\epsilon}_{33}) \tag{12}$$

where

$$C_{11}^* = \frac{1}{2(1-\nu_0)}\left\{-1+6\nu_0-\frac{2}{(\alpha_1^2-1)}+3(1-2\nu_0)g_1\right\}$$
$$+\frac{1}{2(1-\nu_0)}\left(\frac{\mu_1-\mu_0}{\lambda_1-\lambda_0}\right)\left[1+2\nu_0+\left\{1-2\nu_0-\frac{3}{(\alpha_1^2-1)}\right\}g_1\right]+2\left(\frac{\lambda_0+\mu_0}{\lambda_1-\lambda_0}\right)$$

$$C_{12}^* = 1-\left(\frac{1-2\nu_0}{1-\nu_0}\right)g_1+\frac{1}{2(1-\nu_0)}\left(\frac{\mu_1-\mu_0}{\lambda_1-\lambda_0}\right)\left[-\frac{2\alpha_1^2}{(\alpha_1^2-1)}+\left\{\frac{3\alpha_1^2}{(\alpha_1^2-1)}\right.\right.$$
$$\left.\left.-(1-2\nu_0)\right\}g_1\right]+\frac{\lambda^0}{(\lambda_1-\lambda^0)}$$

$$C_{21}^* = \frac{1}{2(1-\nu_0)}\left\{-1+6\nu_0-\frac{2}{(\alpha_1^2-1)}+3(1-2\nu_0)g_1\right\}+\frac{2\lambda_0}{(\lambda_1-\lambda_0)}$$
$$+\frac{2}{(1-\nu_0)}\left(\frac{\mu_1-\mu_0}{\lambda_1-\lambda_0}\right)\left[-(1-2\nu_0)-\frac{1}{(\alpha_1^2-1)}+\left\{1-2\nu_0+\frac{3}{2(\alpha_1^2-1)}\right\}g_1\right]$$

$$C_{22}^* = 1-\left(\frac{1-2\nu_0}{1-\nu_0}\right)g_1+\left(\frac{\lambda_0+2\mu_0'}{\lambda_1-\lambda_0}\right) \tag{13}$$
$$+\frac{1}{(1-\nu_0)}\left(\frac{\mu_1-\mu^0}{\lambda_1-\lambda^0}\right)\left[1-2\nu_0+\frac{(3\alpha_1^2-1)}{(\alpha_1^2-1)}-\left\{1-2\nu_0+\frac{3\alpha_1^2}{(\alpha_1^2-1)}\right\}g_1\right]$$

$$D_1^* = 1+\left(\frac{\mu_1-\mu_0}{\lambda_1-\lambda_0}\right)$$

$$D_2^* = 1+2\left(\frac{\mu_1-\mu_0}{\lambda_1-\lambda_0}\right).$$

In eqn (13) λ, μ are Lamé constants, ν is Poisson's ratio, α is the aspect ratio of the fiber, and g is defined in Appendix B. The subscripts 0 and 1 in the above equation denote the matrix and fiber-1, respectively.

Noting that $\epsilon_{11}^0 = -\nu_0\sigma^0/E_0$ and $\epsilon_{33}^0 = \sigma^0/E_0$ where E_0 is the matrix Young's modulus, we solve for ϵ_{11}^* and ϵ_{33}^* in eqns (11) and (12) to obtain

$$\epsilon_{11}^* = \frac{B_1^*}{A^*}\tilde{\epsilon}_{11}+\frac{B_2^*}{A^*}\tilde{\epsilon}_{33}+\frac{\sigma^0}{E_0}\frac{(B_2^*-\nu_0B_1^*)}{A^*} \tag{14}$$

$$\epsilon_{33}^* = \frac{B_3^*}{A^*}\tilde{\epsilon}_{11}+\frac{B_4^*}{A^*}\tilde{\epsilon}_{33}+\frac{\sigma_0}{E_0}\frac{(B_4^*-\nu_0B_3^*)}{A^*} \tag{15}$$

where

$$\begin{aligned} A^* &= C_{11}^*C_{22}^*-C_{21}^*C_{12}^* \\ B_1^* &= 2(C_{12}^*-D_1^*C_{22}^*) \\ B_2^* &= D_2^*C_{12}^*-C_{22}^* \\ B_3^* &= 2(D_1^*C_{21}^*-C_{11}^*) \\ B_4^* &= C_{21}^*-D_2^*C_{11}^*. \end{aligned} \tag{16}$$

The stress disturbed by Ω_1, σ^1_{ij} is obtained by eqns (4), (5), (14) and (15) as

$$\frac{\sigma^1_{11}}{\mu^0} = \left\{\frac{2}{(1-2\nu_0)} + \frac{(H^*_{11}B_1{}^* + H^*_{12}B_3{}^*)}{A^*}\right\}\tilde{\epsilon}_{11} + \left\{\frac{2\nu_0}{1-2\nu_0} + \frac{(H^*_{11}B_2{}^* + H^*_{12}B_4{}^*)}{A^*}\right\}\tilde{\epsilon}_{33} + \frac{\sigma^0}{E_0}\left\{\frac{H^*_{11}(B_2{}^* - \nu_0 B_1{}^*)}{A^*} + \frac{H^*_{12}(B^*_4 - \nu_0 B^*_3)}{A^*}\right\} \tag{17}$$

$$\frac{\sigma^1_{33}}{\mu^0} = \left\{\frac{4\nu_0}{(1-2\nu_0)} + \frac{(H^*_{21}B_1{}^* + H^*_{22}B_3{}^*)}{A^*}\right\}\tilde{\epsilon}_{11} + \left\{\frac{2(1-2\nu_0)}{(1-2\nu_0)} + \frac{(H^*_{21}B_2{}^* + H^*_{22}B_4{}^*)}{A^*}\right\}\tilde{\epsilon}_{33} + \frac{\sigma^0}{E_0}\left\{\frac{H^*_{21}(B_2{}^* - \nu_0 B_1{}^*)}{A^*} + \frac{H^*_{22}(B_4{}^* - \nu_0 B_3{}^*)}{A^*}\right\} \tag{18}$$

where

$$H^*_{11} = 2\left\{\frac{2\nu_0}{(1-2\nu_0)}(S^1_{1111} + S^1_{1122} + S^1_{3311} - 1) + S^1_{1111} + S^1_{1122} - 1\right\} \tag{19}$$

$$H^*_{12} = \frac{2\nu_0}{(1-2\nu_0)}(2S^1_{1133} + S^1_{3333} - 1) + 2S^1_{1133}$$

$$H^*_{21} = \frac{4\nu_0}{(1-2\nu_0)}(S^1_{1111} + S^1_{1122} + S^1_{3311} - 1) + 4S^1_{3311}$$

$$H^*_{22} = \frac{2\nu_0}{(1-2\nu_0)}(2S^1_{1133} + S^1_{3333} - 1) + 2(S^1_{3333} - 1)$$

and S^1_{ijkl} is the Eshelby's tensor for fiber-1 and is given in Appendix B.

Next we solve for ϵ^{**}_{11} and ϵ^{**}_{33} in eqns (6) and (8), and for σ^2_{11} and σ^2_{33} in eqn (7). The solution procedure is the same as in the foregoing steps except for eqns (13) and (19), which will be obtained below, depending on the type of inhomogeneity Ω_2.

(*a*) Ω_2 denotes fiber phase. The solutions corresponding to eqns (14) and (15) are

$$\epsilon^{**}_{11} = \frac{B^{**}_1}{A^{**}}\tilde{\epsilon}_{11} + \frac{B^{**}_2}{A^{**}}\tilde{\epsilon}_{33} + \frac{\sigma^0}{E_0}\frac{(B^{**}_2 - \nu_0 B^{**}_1)}{A^{**}} \tag{20}$$

$$\epsilon^{**}_{33} = \frac{B^{**}_3}{A^{**}}\tilde{\epsilon}_{11} + \frac{B^{**}_4}{A^{**}}\tilde{\epsilon}_{33} + \frac{\sigma^0}{E_0}\frac{(B^{**}_4 - \nu_0 B^{**}_3)}{A^{**}} \tag{21}$$

where the superscript ** denotes the inhomogeneity Ω_2. In the above equations A^{**}, B^{**}_1, B^{**}_2, B^{**}_3 and B^{**}_4 are given by eqn (16) where C^*_{ij} and D^*_i are replaced by C^{**}_{ij} and D^{**}_i respectively.

The coefficients C^{**}_{ij} and D^{**}_i are of the same forms as those given by eqn (17) except that the superscript * and the subscript 1 are replaced by ** and 2, respectively.

Likewise, the disturbed stress σ_{ij}^2 is given by

$$\frac{\sigma_{11}^2}{\mu_0}=\left\{\frac{2}{(1-2\nu_0)}+\frac{(H_{11}^{**}B_1^{**}+H_{12}^{**}B_3^{**})}{A^{**}}\right\}\tilde{\epsilon}_{11}+\left\{\frac{2\nu_0}{(1-2\nu_0)}+\frac{(H_{11}^{**}B_2^{**}+H_{12}^{**}B_4^{**})}{A^{**}}\right\}\tilde{\epsilon}_{33}+\frac{\sigma_0}{E_0}\left\{\frac{H_{11}^{**}(B_2^{**}-\nu_0B_1^{**})}{A^{**}}+\frac{H_{12}^{**}(B_4^{**}-\nu_0B_3^{**})}{A^{**}}\right\} \tag{22}$$

$$\frac{\sigma_{33}^2}{\mu_0}=\left\{\frac{4\nu_0}{(1-2\nu_0)}+\frac{(H_{21}^{**}B_1^{**}+H_{22}^{**}B_3^{**})}{A^{**}}\right\}\tilde{\epsilon}_{11}+\left\{\frac{2(1-\nu_0)}{(1-2\nu_0)}+\frac{(H_{21}^{*}B_2^{**}+H_{22}^{**}B_4^{*})}{A^{**}}\right\}\tilde{\epsilon}_{33}+\frac{\sigma^0}{E_0}\left\{\frac{H_{21}^{**}(B_2^{**}-\nu_0B_1^{**})}{A^{**}}+\frac{H_{22}^{**}(B_4^{**}-\nu_0B_3^{**})}{A^{**}}\right\} \tag{23}$$

where H_{ij}^{**} is given by eqn (19) and S_{ijkl}^1 is replaced by S_{ijkl}^2 which is given in Appendix B.

(b) Ω_2 denotes partuculate phase. A particulate filler is assumed to be of spherical shape. The expressions for ϵ_{ij}^{**} and σ_{ij}^2 are again given by eqns (20)–(23) except that the coefficients C_{ij}^{**} and H_{ij}^{**} are now given by

$$\begin{aligned}
C_{11}^{**}&=\frac{2(1+\nu_0)}{3(1-\nu_0)}+\frac{4}{5(1-\nu_0)}\left(\frac{\mu_2-\mu_0}{\lambda_2-\lambda_0}\right)+2\left(\frac{\lambda_0+\mu_0}{\lambda_2-\lambda_0}\right)\\
C_{12}^{**}&=\frac{(1+\nu_0)}{3(1-\nu_0)}-\frac{2(1-5\nu_0)}{15(1-\nu_0)}\left(\frac{\mu_2-\mu_0}{\lambda_2-\lambda_0}\right)+\frac{\lambda_0}{\lambda_2-\lambda_0}\\
C_{21}^{**}&=C_{12}^{**}\\
C_{22}^{**}&=\frac{(1+\nu_0)}{3(1-\nu_0)}+\frac{2(7-5\nu_0)}{15(1-\nu_0)}\left(\frac{\mu_2-\mu_0}{\lambda_2-\lambda_0}\right)+\left(\frac{\lambda_0+2\mu_0}{\lambda_2-\lambda_0}\right)
\end{aligned} \tag{24}$$

$$\begin{aligned}
H_{11}^{**}&=-\frac{2(9+5\nu_0)}{15(1-\nu_0)}\\
H_{12}^{**}&=-\frac{2(1+5\nu_0)}{15(1-\nu_0)}\\
H_{21}^{**}&=2H_{12}^{**}\\
H_{22}^{**}&=-\frac{16}{15(1-\nu_0)}
\end{aligned} \tag{25}$$

From eqn (1), we have

$$\frac{\langle\sigma_{11}\rangle_M}{\mu_0}=\frac{2}{(1-2\nu_0)}\tilde{\epsilon}_{11}+\frac{2\nu_0}{(1-2\nu_0)}\tilde{\epsilon}_{33} \tag{26}$$

$$\frac{\langle\sigma_{33}\rangle_M}{\mu_0}=\frac{4\nu_0}{(1-2\nu_0)}\tilde{\epsilon}_{11}+\frac{2(1-\nu_0)}{(1-2\nu_0)}\tilde{\epsilon}_{33}. \tag{27}$$

After having expressed $\langle\sigma_{ij}\rangle_M$, $\langle\sigma_{ij}^1\rangle$ and $\langle\sigma_{ij}^2\rangle$ in terms of $\tilde{\epsilon}_{ij}$ we substitute them into eqn (9) to solve for $\tilde{\epsilon}_{11}$ and $\tilde{\epsilon}_{33}$:

$$\tilde{\epsilon}_{11}=\frac{S_1}{S}\left(\frac{\sigma^0}{E_0}\right) \tag{28}$$

$$\tilde{\epsilon}_{33}=\frac{S_2}{S}\left(\frac{\sigma^0}{E_0}\right) \tag{29}$$

where S, S_1 and S_2 are given in Appendix C. With eqns (28) and (29), we can obtain eigenstrains ϵ_{ij}^{*} and ϵ_{ij}^{**} as

$$\epsilon_{11}^{*} = \left\{\frac{(B_1{}^*S_1 + B_2{}^*S_2)}{A^*S} + \frac{(B_2{}^* - \nu_0 B_1{}^*)}{A^*}\right\}\left(\frac{\sigma^0}{E_0}\right) \tag{30}$$

$$\epsilon_{33}^{*} = \left\{\frac{(B_3{}^*S_1 + B_4{}^*S_2)}{A^*S} + \frac{(B_4{}^* - \nu_0 B_3{}^*)}{A^*}\right\}\left(\frac{\sigma^0}{E_0}\right) \tag{31}$$

$$\epsilon_{11}^{**} = \left\{\frac{(B_1^{**}S_1 + B_2^{**}S_2)}{A^{**}S} + \frac{(B_2^{**} - \nu_0 B_1^{**})}{A^{**}}\right\}\left(\frac{\sigma^0}{E_0}\right) \tag{32}$$

$$\epsilon_{33}^{**} = \left\{\frac{(B_3^{**}S_1 + B_4^{**}S_2)}{A^{**}S} + \frac{(B_4^{**} - \nu_0 B_3^{**})}{A^{**}}\right\}\left(\frac{\sigma^0}{E_0}\right) \tag{33}$$

2.2 *Computation of longitudinal Young's modulus* E_L *of a hybrid composite*

When all fibers are aligned in the uniaxial loading direction, the equation of the equivalence of strain energy (eqn (10)) is reduced to

$$\frac{\sigma^{0^2}}{2E_L} = \frac{\sigma^{0^2}}{2E_0} + \frac{\sigma^0\epsilon_{33}^{*}f_1}{2} + \frac{\sigma^0\epsilon_{33}^{**}f_2}{2}. \tag{34}$$

With eqns (31) and (33), eqn (34) provides us with the longitudinal Young's modulus E_L of the hybrid composite:

$$\frac{E_L}{E_0} = \frac{1}{1+\eta} \tag{35}$$

where

$$\eta = f_1\left\{\frac{(B_3^{*}S_1 + B_4^{*}S_2)}{A^*S} + \frac{(B_4^{*} - \nu_0 B_3^{*})}{A^*}\right\} + f_2\left\{\frac{(B_3^{**}S_1 + B_4^{**}S_2)}{A^{**}S} + \frac{(B_4^{**} - \nu_0 B_3^{**})}{A^{**}}\right\}. \tag{36}$$

It should be noted that S, S_1 and S_2 are also functions of f_1 and f_2 (see Appendix C). When the two kinds of fillers are identical, i.e. $\Omega_1 = \Omega_2$ and the volume fraction of the filler is small, one can easily obtain the results based on Eshelby's equivalent inclusion method (without back stress analysis).

3. RESULTS AND DISCUSSION

Since a hybrid composite consists of three phases, a number of parameters characteristic of the three phases must be specified in order to compute E_L. Here we set Poisson's ratio of the matrix as 0.35 and those of two kinds of fillers as 0.3 throughout our computation. As a demonstration of our results, the following cases are computed:

(i) Case-1: Compute E_L/E_0 for given stiffnesses of the fillers, $(E/E_0)_1 = 50$ and $(E/E_0)_2 = 100$, and given ratio of the volume fraction of the fillers, $f_2/f_1 = 3$ for various aspect ratios of the fibers.

(ii) Case-2: Compute E_L/E_0 of a fiber-particulate composite for $(E/E_0)_1 = 50$, $(E/E_0)_2 = 50$ and $f_1 = f_2$.

The results of Case-1 are shown as solid curves in Fig. 2 where E_L/E_0 computed by a "rule of mixtures" which can be obtained asymptotically by increasing the aspect ratios of fibers, is shown as a dotted line. It is noted in Fig. 2 that the aspect ratio l/d being 1 corresponds to the case of a particulate filler.

The results of Case-2 are shown in Fig. 3 where two extreme cases are also investigated, i.e. two kinds of fillers are identical and they are either of the fiber type or of the particulate type. It is concluded from Fig. 3 that "a volume average approach" to combine the results of

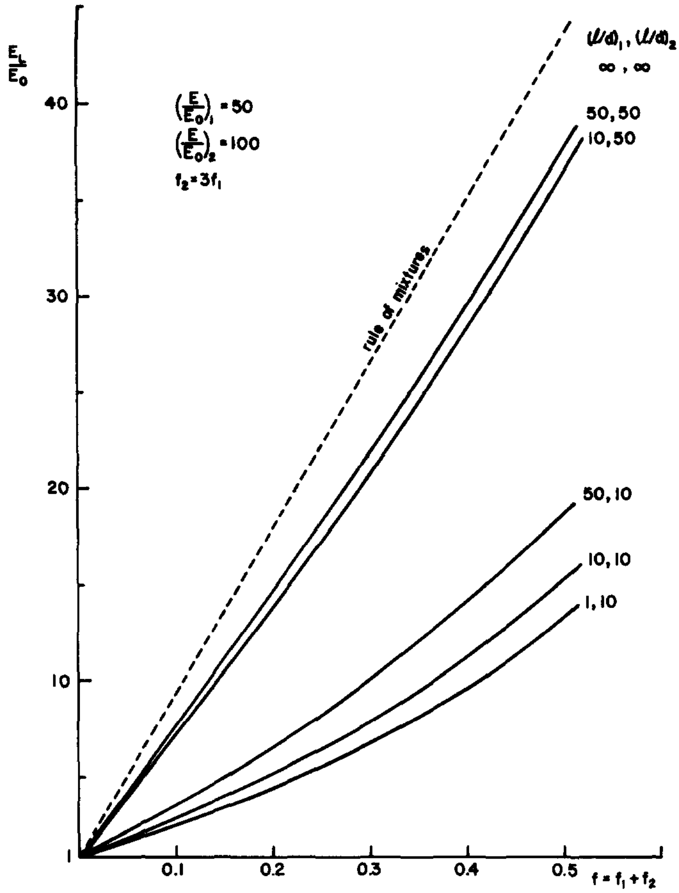

Fig. 2. Longitudinal Young's modulus of a hybrid composite vs. volume fraction of fillers with $f_2 = 3f_1$.

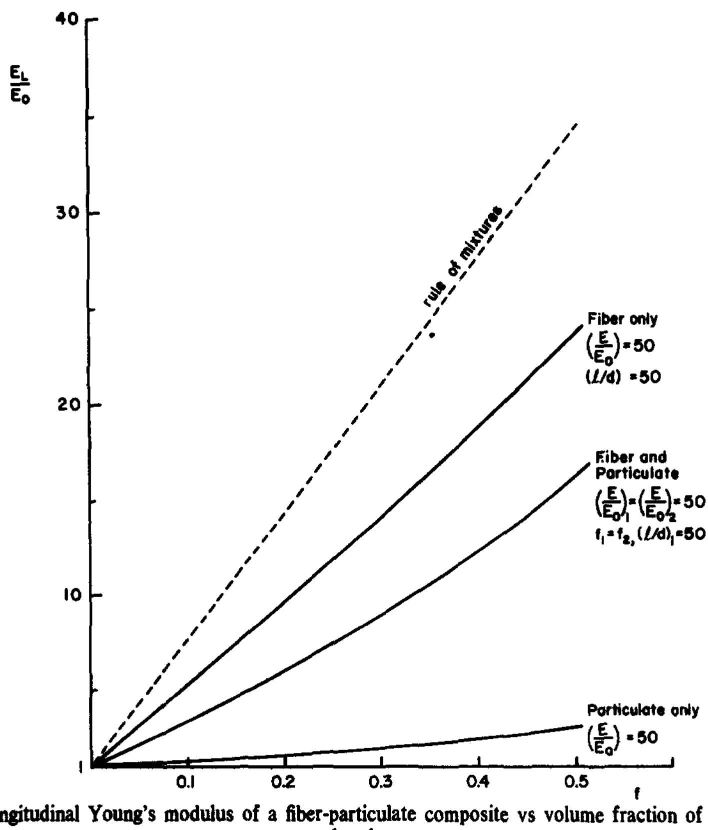

Fig. 3. Longitudinal Young's modulus of a fiber-particulate composite vs volume fraction of fillers with $f_1 = f_2$.

fiber only and particulate only does not provide the results of fiber-particulate system. In other words, if it does, then the curve of fiber-particulate system would have been located exactly in the middle between those cases of fiber only and particulate only since $f_1 = f_2$. Also the results based on a "rule of mixtures" are plotted as a dotted line in Fig. 3.

Finally it should be noted that the present formulation can be easily extended to the case of more than three-phase materials since all the interactions among various kinds of inhomogeneity are carried by $\bar{\epsilon}_{ij}$ (see eqns (2) and (6)). Hence in the equation of the equivalence of the strain energies (eqn (10)), an addition of another kind of inhomogeneity simply leads to that of another term carrying the corresponding eigenstrain to the r.h.s. of eqn (10).

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APPENDIX A

A strain energy W of a hybrid composite containing two kinds of inhomogeneities (Ω_1 and Ω_2) is given by

$$W = \frac{1}{2}\int_D (\sigma_{ij}^0 + \sigma_{ij})(u_{i,j}^0 + \bar{u}_{i,j} + u_{i,j})\,\mathrm{d}V \tag{A1}$$

where the domain of integration is D (entire body) including the matrix, Ω_1 and Ω_2, σ_{ij} and $u_{i,j}$ are the disturbances of the stress and strain (displacement gradient) due to inhomogeneities and $\bar{u}_i$ is the average displacement in the matrix and the symmetric component of its gradient is $\bar{\epsilon}_{ij}$ (see eqn (1)). We expand the integrand in eqn (A1) as

$$(\sigma_{ij}^0 + \sigma_{ij})(u_{i,j}^0 + \bar{u}_{i,j} + u_{i,j}) = \sigma_{ij}^0 u_{i,j}^0 + \sigma_{ij}^0(\bar{u}_{i,j} + u_{i,j}) + \sigma_{ij}(u_{i,j}^0 + \bar{u}_{i,j} + u_{i,j}) \tag{A2}$$

Note that

$$\int_D \sigma_{ij}(u_{i,j}^0 + \bar{u}_{i,j} + u_{i,j})\,\mathrm{d}V = -\int_D \sigma_{ij,j}(u_i^0 + \bar{u}_i + u_i)\,\mathrm{d}V + \int_{|D|} \sigma_{ij}n_j(u_i^0 + \bar{u}_i + u_i)\,\mathrm{d}S = 0, \tag{A3}$$

since $\sigma_{ij,j} = 0$ in D and $\sigma_{ij}n_j = 0$ on $|D|$, where $|D|$ is the boundary of D. In the derivation of (A3) Gauss' divergence theorem has been used.

Next consider the second term on the right hand side of eqn (A2). This term can be rewritten as in terms of eigenstrain e_{ij}^* dfined in Ω.

$$\sigma_{ij}^0(\bar{u}_{i,j} + u_{i,j}) = \sigma_{ij}^0\{(\bar{u}_{i,j} + u_{i,j} - e_{ij}^*) + e_{ij}^*\} = \sigma_{ij}u_{i,j}^0 + \sigma_{ij}^0 e_{ij}^* \tag{A4}$$

where the Eshelby's equivalent inclusion method is applied;

$$\sigma_{ij} = C_{ijkl}^0(\bar{u}_{k,l} + u_{k,l} - e_{kl}^*) \quad \text{in } D. \tag{A5}$$

The first term on the right hand side of eqn (A4) vanishes upon integration over D by the same token leading to eqn (A3). Thus, a strain energy W (eqn (A1)) is simplified as

$$W = \frac{1}{2}\int_D \sigma_{ij}^0 u_{i,j}^0\,\mathrm{d}V + \frac{1}{2}\int_D \sigma_{ij}^0 e_{ij}^*\,\mathrm{d}V \tag{A6}$$

Since we have two kinds of eigenstrains ϵ_{ij}^* and ϵ_{ij}^{**}, e_{ij}^* is given by

$$e_{ij}^* = \begin{cases} \epsilon_{ij}^* & \text{in } \Omega_1 \\ \epsilon_{ij}^{**} & \text{in } \Omega_2 \\ 0 & \text{in } D\text{-}\Omega_1\text{-}\Omega_2. \end{cases} \tag{A7}$$

With eqn (A7), eqn (A6) is further reduced to

$$W = \frac{1}{2}\int_D \sigma_{ij}^0 u_{i,j}^0\,\mathrm{D}V + \frac{1}{2}\int_{\Omega_1} \sigma_{ij}^0 \epsilon_{ij}^*\,\mathrm{d}V + \frac{1}{2}\int_{\Omega_2} \sigma_{ij}^0 \epsilon_{ij}^{**}\,\mathrm{d}V. \tag{A8}$$

A strain energy W can be interpreted as per a unit volume. Then, we obtain from (A8)

$$W = \frac{1}{2}\sigma_{ij}^0 u_{i,j}^0 + \frac{1}{2} f_1 \sigma_{ij}^0 \epsilon_{ij}^{*} + \frac{1}{2} f_2 \sigma_{ij}^0 \epsilon_{ij}^{**}. \tag{A9}$$

Under the applied stress σ_{ij}^0, a composite has a strain energy $1/2 C_{ijkl}^{c\,-1} \sigma_{ij}^0 \sigma_{kl}^0$ where $C_{ijkl}^{c\,-1}$ is the compliance of the composite. Thus we can obtain the equation of the equivalence of strain energy (eqn (10)).

APPENDIX B SHELBY'S TENSORS

For a fiber-like inclusion, S_{ijkl}^1 are given by

$$\begin{aligned}
S_{1111}^1 = S_{2222}^1 &= \frac{3}{8(1-\nu_0)}\frac{\alpha^2}{(\alpha^2-1)} + \frac{1}{4(1-\nu_0)}\left\{1 - 2\nu_0 - \frac{9}{4(\alpha^2-1)}\right\}g \\
S_{3333}^1 &= \frac{1}{2(1-\nu_0)}\left[1 - 2\nu_0 + \frac{(3\alpha^2-1)}{(\alpha^2-1)} - \left\{1 - 2\nu_0 + \frac{3\alpha^2}{(\alpha^2-1)}\right\}g\right] \\
S_{1122}^1 = S_{2211}^1 &= \frac{1}{4(1-\nu_0)}\left\{\frac{\alpha^2}{2(\alpha^2-1)} - (1-2\nu_0) - \frac{3}{4(\alpha^2-1)}g\right\} \\
S_{1133}^1 = S_{2233}^1 &= -\frac{1}{2(1-\nu_0)}\frac{\alpha^2}{(\alpha^2-1)} + \frac{1}{4(1-\nu_0)}\left\{\frac{3\alpha^2}{(\alpha^2-1)} - (1-2\nu_0)\right\}g \\
S_{3311}^1 = S_{3322}^1 &= -\frac{1}{2(1-\nu_0)}\left\{1 - 2\nu_0 + \frac{1}{(\alpha^2-1)}\right\} + \frac{1}{2(1-\nu_0)}\left\{1 - 2\nu_0 + \frac{3}{2(\alpha^2-1)}\right\}g
\end{aligned} \tag{B1}$$

where ν_0 is Poisson's ratio of a matrix, α is aspect ratio of a fiber ($= l/d$), and g is given by

$$g = \frac{\alpha}{(\alpha^2-1)^{3/2}}\{\alpha(\alpha^2-1)^{1/2} - \cosh^{-1}\alpha\}. \tag{B2}$$

For spherical inclusion, non vanishing S_{ijkl}^2 are

$$\begin{aligned}
S_{1111}^2 = S_{2222}^2 = S_{3333}^2 &= \frac{(7-5\nu_0)}{15(1-\nu_0)} \\
S_{1122}^2 = S_{2233}^2 = S_{3311}^2 &= -\frac{(1-5\nu_0)}{15(1-\nu_0)} \\
S_{1212}^2 = S_{2323}^2 = S_{3131}^2 &= \frac{(4-5\nu_0)}{15(1-\nu_0)}.
\end{aligned} \tag{B3}$$

APPENDIX C

A substitution of $\langle\sigma_{ij}\rangle_M$, $\langle\sigma_{ij}^1\rangle$ and $\langle\sigma_{ij}^2\rangle$ into eqn (9) yields the value of S, S_1 and S_2 defined in eqns (28) and (29) as

$$\begin{aligned}
S &= Q_{11}Q_{22} - Q_{21}Q_{12} \\
S_1 &= Q_{12}R_2 - Q_{22}R_1 \\
S_2 &= Q_{21}R_1 - Q_{11}R_2
\end{aligned} \tag{C1}$$

where

$$\begin{aligned}
Q_{11} &= \frac{2(1-f_1-f_2)}{(1-2\nu_0)} + f_1\left\{\frac{2}{(1-2\nu_0)} + \frac{(H_{11}^{*}B_1^{*} + H_{12}^{*}B_3^{*})}{A^{*}}\right\} + f_2\left\{\frac{2}{(1-2\nu_0)} + \frac{(H_{11}^{**}B_1^{**} + H_{12}^{**}B_3^{**})}{A^{**}}\right\} \\
Q_{12} &= \frac{2\nu_0(1-f_1-f_2)}{(1-2\nu_0)} + f_1\left\{\frac{2\nu_0}{(1-2\nu_0)} + \frac{(H_{11}^{*}B_2^{*} + H_{12}^{*}B_4^{*})}{A^{*}}\right\} + f_2\left\{\frac{2\nu_0}{(1-2\nu_0)} + \frac{(H_{11}^{**}B_2^{**} + H_{12}^{**}B_4^{**})}{A^{**}}\right\} \\
Q_{21} &= \frac{4\nu_0(1-f_1-f_2)}{(1-2\nu_0)} + f_1\left\{\frac{4\nu_0}{(1-2\nu_0)} + \frac{(H_{21}^{*}B_1^{*} + H_{22}^{*}B_3^{*})}{A^{*}}\right\} + f_2\left\{\frac{4\nu_0}{(1-2\nu_0)} + \frac{(H_{21}^{**}B_1^{**} + H_{22}^{**}B_3^{**})}{A^{**}}\right\}
\end{aligned} \tag{C2}$$

$$Q_{22}=\frac{2(1-\nu_0)(1-f_1-f_2)}{(1-2\nu_0)}+f_1\left\{\frac{2(1-\nu_0)}{(1-2\nu_0)}+\frac{(H^*_{21}B_2{}^*+H^*_{22}B_4{}^*)}{A^*}\right\}+f_2\left\{\frac{2(1-\nu_0)}{(1-2\nu_0)}+\frac{(H^{**}_{21}B^{**}_2+H^{**}_{22}B^{**}_4)}{A^{**}}\right\}$$

$$R_1=\frac{f_1}{A^*}\{H^*_{11}(B_2{}^*-\nu_0B_1{}^*)+H^*_{12}(B_4{}^*-\nu_0B_3{}^*)\}+\frac{f_2}{A^{**}}\{H^{**}_{11}(B^{**}_2-\nu_0B^{**}_1)+H^{**}_{12}(B^{**}_4-\nu_0B^{**}_3)\}$$

$$R_2=\frac{f_1}{A^*}\{H^*_{21}(B_2{}^*-\nu_0B_1{}^*)+H^*_{22}(B_4{}^*-\nu_0B_3{}^*)\}+\frac{f_2}{A^{**}}\{H^{**}_{21}(B^{**}_2-\nu_0B^{**}_1)+H^{**}_{22}(B^{**}_4-\nu_0B^{**}_3)\}.$$